

AVERAGE MIXED VOLUME UNDER PROJECTION

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ABSTRACT. The average mixed volume of a random projection of d convex bodies in \mathbb{R}^n is bounded above in terms of a quermassintegral.

1. INTRODUCTION

The *mixed volume* of the n -tuple of convex bodies $(\mathcal{A}_1, \dots, \mathcal{A}_n)$, $\mathcal{A}_i \subset \mathbb{R}^n$ is defined as

$$V(\mathcal{A}_1, \dots, \mathcal{A}_n) \stackrel{\text{def}}{=} \frac{1}{n!} \frac{\partial^n}{\partial t_1 \partial t_2 \dots \partial t_n} \text{Vol}(t_1 \mathcal{A}_1 + \dots + t_n \mathcal{A}_n)$$

where $t_1, \dots, t_n \geq 0$ and the derivative is taken at $t = 0$. The normalization factor $1/n!$ ensures that $V(\mathcal{A}, \dots, \mathcal{A}) = \text{Vol}(\mathcal{A})$. In that sense the mixed volume generalizes the usual volume.

The definition above is still valid if we allow the \mathcal{A}_i to have empty volume, so in this paper we will only require the \mathcal{A}_i to be closed convex sets.

The mixed volume was introduced by Minkowski (1901) for $n = 3$. If $\mathcal{A} \subset \mathbb{R}^3$ is a convex body, the Steiner formula

$$\text{Vol}(\mathcal{A} + \epsilon B^3) = \text{Vol}(\mathcal{A}) + S\epsilon + \pi B\epsilon^2 + \frac{4}{3}\pi\epsilon^3$$

implies that

$$V(\mathcal{A}, \mathcal{A}, B^3) = 3S \quad \text{and} \quad V(\mathcal{A}, B^3, B^3) = 3\pi B$$

with S is the total area of $\partial\mathcal{A}$ and B its total mean curvature (assuming $\partial\mathcal{A}$ is smooth). The quantities $V(\mathcal{A}, \dots, \mathcal{A}, B^n, \dots, B^n)$ are known as *quermassintegrals*. There is a vast literature on recovering quantities such as the area of $\partial\mathcal{A}$ from the area of projections of \mathcal{A} .

However, most of the literature deals with mixed volumes for two different convex bodies, one being the ball B^n . Mixed volumes with many possibly different convex bodies seem to arise mostly in connection with counting zeros of sparse polynomial systems or obtaining a starting system for path-following. The main result in this note arised during the complexity analysis of such an algorithm.

Let $G(d, n)$ denote the Grassmannian manifold of d -dimensional subspaces of \mathbb{R}^n . It is endowed with the unique probability measure invariant under the orthogonal group $O(n)$. In this note we prove the following result:

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Theorem 1. *Let $\mathcal{A}_1, \dots, \mathcal{A}_d$ be compact convex sets in \mathbb{R}^n . Then,*

$$\operatorname{Avg}_{P \in G(d, n)} V(P(\mathcal{A}_1), \dots, P(\mathcal{A}_d)) \leq \frac{\operatorname{Vol}(B^d)}{\operatorname{Vol}(B^n)} V(\mathcal{A}_1, \dots, \mathcal{A}_d, B^n, \dots, B^n).$$

By replacing \mathcal{A}_i with B^n and $P(\mathcal{A}_i)$ by B^d , one quickly checks that the bound above is sharp.

2. PROOF OF THE THEOREM

Before proving theorem 1, we will restate it in a more convenient form. If $Q \in O(n)$, let q_j denote the j -th row of Q . If Q is uniformly distributed in $O(n)$ with respect to the Haar measure, the projection $P : \mathbb{R}^n \rightarrow \mathbb{R}^d$,

$$P : x \mapsto \begin{bmatrix} q_1 x \\ \vdots \\ q_d x \end{bmatrix}$$

is uniformly distributed in $G(d, n)$. We have

$$d! V(P(\mathcal{A}_1), \dots, P(\mathcal{A}_d)) = n! V(\mathcal{A}_1, \dots, \mathcal{A}_d, [0, q_{d+1}], \dots, [0, q_n]).$$

Therefore, what we need to prove is the following restatement of Theorem 1:

Theorem 1-A. *Let $\mathcal{A}_1, \dots, \mathcal{A}_d$ be compact convex sets in \mathbb{R}^n . Then,*

$$\begin{aligned} \operatorname{Avg}_{Q \in O(n)} V(\mathcal{A}_1, \dots, \mathcal{A}_d, [0, q_{d+1}], \dots, [0, q_{d+n}]) &\leq \\ &\leq \frac{d! \operatorname{Vol}(B^d)}{n! \operatorname{Vol}(B^n)} V(\mathcal{A}_1, \dots, \mathcal{A}_d, B^n, \dots, B^n). \end{aligned}$$

Remark 2. Explicitly,

$$\frac{d! \operatorname{Vol}(B^d)}{n! \operatorname{Vol}(B^n)} = \frac{1}{(2\sqrt{\pi})^{n-d}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{n+1}{2})}.$$

This is an immediate consequence of the duplication formula (Abramowitz and Stegun, 1964, formula 6.1.18),

$$\Gamma(2z) = \frac{1}{\sqrt{2\pi}} 2^{2z-1/2} \Gamma(z) \Gamma(z + 1/2).$$

Let CONV_n denote the set of all compact convex subsets of \mathbb{R}^n . This set is closed under Minkowski sum and under multiplication by non-negative real numbers. In that sense, CONV_n is an ‘algebra’ over the non-negative real numbers. We recall the following definitions:

Definition 3. A function $F : \operatorname{CONV}_n \rightarrow \mathbb{R}^+$ is linear if for all $t_1, t_2 \in \mathbb{R}^+$ and for all $\mathcal{A}_1, \mathcal{A}_2 \in \operatorname{CONV}_n$, $F(t_1 \mathcal{A}_1 + t_2 \mathcal{A}_2) = t_1 F(\mathcal{A}_1) + t_2 F(\mathcal{A}_2)$.

Definition 4. A function $F : \operatorname{CONV}_n \rightarrow \mathbb{R}^+$ is monotone if $\mathcal{A} \subseteq \mathcal{B}$ implies that $F(\mathcal{A}) \leq F(\mathcal{B})$.

The mixed volume $V(\mathcal{A}_1, \dots, \mathcal{A}_n)$ is linear and monotone in each of the variables. The key for proving theorem 1 is the following Lemma, that I could not find it in the literature.

Lemma 5. *Let $F : \text{CONV}_k \rightarrow \mathbb{R}^+$ be linear and monotone. Then*

$$\text{Avg}_{c \in S^{k-1}} (F([0, c])) \leq r_k F(B^k)$$

with $r_k = \frac{\Gamma(k/2)}{\sqrt{\pi}(k-1)\Gamma(\frac{k-1}{2})}$. This constant r_k is the smallest with such a property.

Proof. For every $\epsilon > 0$, there is a closed covering of S^{k-1} into measurable subsets of radius ϵ , say $S^{k-1} = \cup_{\lambda \in \Lambda} V_\lambda$, and such that $V_\lambda \cap V_{\lambda'}$ has measure zero for $\lambda \neq \lambda'$. The sets $W_\lambda = \text{Conv}(\{0\} \cup V_\lambda)$ are a closed covering of B^k with the same property.

$$\begin{aligned} \text{Avg}_{c \in S^{k-1}} (F([0, c])) &= \frac{1}{\text{Vol} S^{k-1}} \int_{S^{k-1}} F([0, c]) \, dS^{k-1}(c) && \text{by definition,} \\ &= \frac{1}{\text{Vol} S^{k-1}} \sum_{\lambda \in \Lambda} \int_{V_\lambda} F([0, c]) \, dS^{k-1}(c) && \text{trivially,} \\ &\leq \frac{1}{\text{Vol} S^{k-1}} \sum_{\lambda \in \Lambda} \int_{V_\lambda} F(W_\lambda) \, dS^{k-1}(c) && \text{by monotonicity,} \\ &\leq \frac{1}{\text{Vol} S^{k-1}} \sum_{\lambda \in \Lambda} (\text{Vol} V_\lambda) F(W_\lambda) && \text{trivially,} \\ &\leq \frac{1}{\text{Vol} S^{k-1}} F \left(\sum_{\lambda \in \Lambda} (\text{Vol} V_\lambda) W_\lambda \right) && \text{by linearity} \end{aligned}$$

After an orthogonal change of coordinates, assume that $\rho(\epsilon)e_1$ is a point of largest norm in $\sum_{\lambda \in \Lambda} (\text{Vol} V_\lambda) W_\lambda$. In particular, there are $x_\lambda \in W_\lambda$ such that $\rho(\epsilon)e_1 = \sum_{\lambda \in \Lambda} (\text{Vol} V_\lambda) x_\lambda$. So,

$$\rho(\epsilon) = \sum_{\lambda \in \Lambda} (\text{Vol} V_\lambda) (x_\lambda)_1 \leq \sum_{(x_\lambda)_1 \geq 0} \text{Vol} V_\lambda \frac{(x_\lambda)_1}{\|x_\lambda\|}.$$

Passing to the limit,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \rho(\epsilon) &= \int_{S^{k-1} \cap [y_1 \geq 0]} y_1 \, dS^{k-1}(y) \\ &= \int_{S^{k-1} \cap [y_1 \geq 0]} e_1 \cdot dn(y) && \text{interpreting as a flow,} \\ &= \int_{B^k \cap [y_1 \geq 0]} \text{div}(e_1) \, dB^k(y) - \int_{B^{k-1}} e_1 \cdot dn(y) && \text{by Gauss theorem,} \\ &= 0 - \int_{B^{k-2}} -1 \, dB^{k-1}(y) \\ &= \text{Vol} B^{k-1} \end{aligned}$$

By monotonicity of F and then linearity, we conclude that

$$\text{Avg}_{c \in S^{k-1}} (F([0, c])) \leq \frac{1}{\text{Vol} S^{k-1}} F(\lim_{\epsilon \rightarrow 0} \rho(\epsilon) B^k) \leq r_k F(B^k)$$

with $r_k = \text{Vol}B^{k-1}/\text{Vol}S^{k-1}$. Using the well-known formulas

$$\text{Vol}B^k = \frac{2\pi^{k/2}}{k \Gamma\left(\frac{k}{2}\right)} \quad \text{and} \quad \text{Vol}S^{k-1} = \frac{2\pi^{k/2}}{\Gamma\left(\frac{k}{2}\right)}$$

we deduce that

$$r_k = \frac{\Gamma\left(\frac{k}{2}\right)}{\sqrt{\pi}(k-1) \Gamma\left(\frac{k-1}{2}\right)}.$$

To establish sharpness, we consider the operator $F(X) = V(B^k, \dots, B^k, X)$. Because of the properties of the mixed volume,

$$F([0, c]) = V(B^k \cap c^\perp, \dots, B^k \cap c^\perp, [0, c]).$$

Thus,

$$\text{Avg}_{c \in S^{k-1}} F([0, c]) = F([0, e_k]) = V(B^{k-1} \times \{0\}, \dots, B^{k-1} \times \{0\}, [0, e_k]).$$

This last value is by definition $\frac{1}{k!}$ times the coefficient in $t_1 t_2 \dots t_k$ of

$$\text{Vol}(t_1 + \dots + t_{k-1})B^{k-1} + t_k[0, e_k] = (t_1 + \dots + t_{k-1})^{k-1} t_k \text{Vol}B^{k-1}.$$

Thus,

$$\text{Avg}_{c \in S^{k-1}} F([0, c]) = \frac{1}{k} \text{Vol}B^{k-1}$$

while the predicted value is $r_k \text{Vol}B^k = \frac{\text{Vol}B^{k-1}}{\text{Vol}S^{k-1}} \text{Vol}B^k = \frac{1}{k} \text{Vol}B^{k-1}$. \square

Proof of Theorem 1. The last row q_n of Q is uniformly distributed in S^{n-1} , so

$$\begin{aligned} \text{Avg}_{Q \in SO(n)} (V(\mathcal{A}_1, \dots, \mathcal{A}_d, [0, q_{d+1}], \dots, [0, q_n])) &= \\ &= \text{Avg}_{q_n \in S^{n-1}} \left(\text{Avg} (V(\mathcal{A}_1, \dots, \mathcal{A}_d, [0, q_{d+1}], \dots, [0, q_n])) \right) \end{aligned}$$

where the inner average is taken over orthonormal frames $(q_{d+1}, \dots, q_{n-1})$ that are orthogonal to q_n . By Lemma 5,

$$\begin{aligned} \text{Avg}_{Q \in SO(n)} (V(\mathcal{A}_1, \dots, \mathcal{A}_d, [0, q_{d+1}], \dots, [0, q_n])) &= \\ &= r_n \text{Avg} (V(\mathcal{A}_1, \dots, \mathcal{A}_d, [0, q_{d+1}], \dots, [0, q_{n-1}], B^n)). \end{aligned}$$

Repeating the same argument,

$$\begin{aligned} \text{Avg}_{Q \in SO(n)} V(\mathcal{A}_1, \dots, \mathcal{A}_d, [0, q_{d+1}], \dots, [0, q_n]) & \\ &\leq r_{d+1} r_{d+2} \dots r_n V(\mathcal{A}_1, \dots, \mathcal{A}_d, B^n, \dots, B^n). \end{aligned}$$

Finally,

$$r_{d+1} r_{d+2} \dots r_n = \frac{\Gamma(n/2) \Gamma(d)}{\pi^{\frac{n-d}{2}} \Gamma(d/2) \Gamma(n)}$$

\square

REFERENCES

- Abramowitz, Milton and Irene A. Stegun. 1964. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, vol. 55, For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. MR0167642 (29 #4914)
- Minkowski, Hermann. 1901. *Sur les surfaces convexes fermées*, C.R. Acad.Sci., Paris **132**, 21–24.

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